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Quantum Hall states as matrix Chern-Simons theory

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Abstract: We propose a finite Chern-Simons matrix model on the plane as an effective description of fractional quantum Hall fluids of finite extent. The quantization of the inverse filling fraction and of the quasiparticle number is shown to arise quantum mechanically and to agree with Laughlin theory. We also point out the effective equivalence of this model, and therefore of the quantum Hall system, with the Calogero model.

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1. Introduction

Chern-Simons actions have appeared in various contexts as topological terms in the action for gauge fields in odd (spacetime) dimensions [1]. Their generalization to noncommutative spaces have also been considered both in the star-product formulation [2, 3] and the operator formulation [4], and have received a lot of attention in the context of string and D-brane physics, induced actions, vortex and nonperturbative solutions etc. [5]–[11].

The planar coordinates of quantum particles in the lowest Landau level of a constant magnetic field provide a well-known and natural realization of noncommutative space [12]. This connection had so far not proven to be of much relevance to noncommutative field theory. Recently, however, Susskind proposed that noncommutative Chern-Simons theory on the plane may provide a description of the (fractional) quantum Hall fluid [13], and specifically of the Laughlin states [14]. In this, noncommutative Chern-Simons theory finds a new context and acquires new interest. Connections between D-brane physics and the quantum Hall effect were also previously examined in [15]–[18].

Chern-Simons theory on the plane necessarily describes a spatially infinite quantum Hall system. The space noncommutativity condition requires an infinite dimensional Hilbert space, and the fields of the theory become operators in this space. It is, however, of interest to also describe finite systems, of limited spatial extent with a finite number of electrons. Such systems form quantum Hall 'droplet' states and exhibit boundary excitations.

The purpose of this paper is to provide such a description. An appropriate finite matrix model will be proposed and shown to reproduce the basic features of the quantum Hall states. The quantization of the model will be performed and shown to imply a quantization of the filling fraction, similar to the quantization of the planar noncommutative Chern-Simons theory [19] (see also [20]), which in turn implies that the underlying particles (electrons) obey integral statistics. Further, the charge of elementary excitations inside the fluid (quasiholes) will be calculated and shown to be quantized in units of the filling fraction, in accordance with Laughlin theory. It will also be pointed out that this model is essentially equivalent to the Calogero model [21]–[25] a well-known integrable system whose connection to fractional statistics [26, 27] and anyons [28]–[31] has been established in different contexts. Finally, some remarks on the expected phase transition of the quantum Hall state to a Wigner crystal will be offered.

2. Chern-Simons theory on the noncommutative plane and quantum Hall states

2.1 Review of Chern-Simons theory

We briefly review the basic features of Chern-Simons (CS) theory on a noncommutative plane and a commutative time and its connection with quantum Hall states [13].

The system to be described consists of $N \to \infty$ electrons on the plane in an external constant magnetic field B (we take their charge $e = 1$). Their coordinates are globally parametrized in a fuzzy way by two (infinite) hermitian 'matrices' X_a, $a = 1, 2$, that is, by two operators on an infinite Hilbert space. Their average density is $\rho_0 = 1/2\pi\theta$. The action is the analog of the gauge action of particles in a magnetic field:

$$S = \int dt \, \frac{B}{2} \operatorname{Tr} \left\{ \epsilon_{ab} (\dot{X}_a + i[A_0, X_a]) X_b + 2\theta A_0 \right\} \tag{2.1}$$

with Tr representing (matrix) trace over the Hilbert space and $[\,.\,,\,.\,]$ representing matrix commutators. The above has the form of a noncommutative CS action, which in operator form reads [4]

$$S = \lambda \int dt \, 2\pi\theta \operatorname{Tr} \left\{ \frac{2}{3} \epsilon_{\mu\nu\rho} D_\mu D_\nu D_\rho + \frac{2}{\theta} A_0 \right\} . \tag{2.2}$$

The 'coordinates' X_a, are the analogs of the covariant derivative operators, $X_a \sim \theta D_a$, and $D_0 = -i\partial_t + A_0$. The time component of the gauge field ensures gauge invariance, its equation of motion imposing the Gauss law constraint

$$[X_1, X_2] = i\theta \,. \tag{2.3}$$

The term in the action linear in A_0 is crucial to ensure that the noncommutative vacuum (2.3) is a solution of the equations, with the noncommutativity parameter $1/\theta$ being essentially the particle density. The coefficient of the CS term λ relates to B as

$$\lambda = \frac{B\theta}{4\pi} = \frac{1}{4\pi\nu} \tag{2.4}$$

with $\nu = 2\pi\rho_0/B$ being the filling fraction.

Gauge transformations are conjugations of X_a or D_a by arbitrary time-dependent unitary operators. In the quantum Hall context they take the meaning of reshuffling the electrons. Equivalently, the X_a can be considered as coordinates of a two-dimensional fuzzy membrane, $2\pi\theta$ playing the role of an area quantum and gauge transformations realizing area preserving diffeomorphisms. The canonical conjugate of X_1 is $P_2 = BX_2$, and the generator of gauge transformations is

$$G = -iB[X_1, X_2] = B\theta = \frac{1}{\nu} \tag{2.5}$$

by virtue of (2.3). Since gauge transformations are interpreted as reshufflings of particles, the above has the interpretation of endowing the particles with quantum statistics of order $1/\nu$.

2.2 Quasiparticle and quasihole states

The classical equation (2.3) has a unique solution, modulo gauge (unitary) transformations, namely the unique irreducible representation of the Heisenberg algebra. So the classical theory has a unique state, the vacuum. Deviations from the vacuum (2.3) can be achieved by introducing sources in the action [13]. A localized source at the origin has a density of the form $\rho = \rho_0 - q\delta^2(x)$ in the continuous (commutative) case, representing a point source of particle number $-q$, that is, a hole of charge q for $q > 0$. The noncommutative analog of such a density is

$$[X_1, X_2] = i\theta(1 + q|0\rangle\langle 0|) \,, \tag{2.6}$$

where $|n\rangle$, $n = 0, 1, \dots$ is an oscillator basis for the (matrix) Hilbert space, $|0\rangle$ representing a state of minimal spread at the origin. In the membrane picture the right-hand side of (2.6) corresponds to area and implies that the area quantum at the origin has been increased to $2\pi\theta(1 + q)$, therefore piercing a hole of area $A = 2\pi\theta q$ and creating a particle deficit $q = \rho_0 A$. We shall call this a quasihole state. (In [13] these were called quasiparticle states; we shall use the opposite terminology here to conform with the standard definition of particle number.) For $q > 0$ we find the quasihole solution of (2.6) as

$$X_1 + iX_2 = \sqrt{2\theta} \sum_{n=1}^{\infty} \sqrt{n+q}\, |n-1\rangle\langle n| \,, \tag{2.7}$$

as was also found in [13].

The case of quasiparticles, $q < 0$ is more interesting. Clearly the area quantum cannot be diminished below zero, and equations (2.6) and (2.7) cannot hold for $-q > 1$. The correct equation is, instead,

$$[X_1, X_2] = i\theta \left(1 - \sum_{n=0}^{k-1} |n\rangle\langle n| - \epsilon |k\rangle\langle k| \right) , \tag{2.8}$$

where we k and ϵ are the integer and fractional part of the quasiparticle charge $-q$. The solution of (2.8) is

$$X_1 + iX_2 = \sum_{n=0}^{k-1} z_n |n\rangle\langle n| + \sqrt{2\theta} \sum_{n=k}^{\infty} \sqrt{n-\epsilon}\, |n-1\rangle\langle n| \,. \tag{2.9}$$

(For $k = 0$ the first sum drops and the second sum starts from $n = 1$.) In the membrane picture, k quanta of the membrane have 'peeled' and occupy positions $z_n = x_n + iy_n$ on the plane, while the rest of the membrane has a deficit of area at the origin equal to $2\pi\theta\epsilon$, leading to a charge surplus ϵ. Clearly the quanta are electrons that sit on top of the continuous charge distribution. If we want all charge density to be concentrated at the origin, we must choose all $z_n = 0$.

We note that the above quasiparticle states for integer q are basically the same as the noncommutative solitons and flux tubes found in noncommutative gauge theory [32]–[35], while the quasihole states have no analog.

Laughlin theory predicts that 'quasihole' excitations in the quantum Hall state have their charge $-q$ quantized in integer units of ν, $q = m\nu$, with m a positive integer. We see that the above discussion gives no hint of this quantization, while we see at least some indication of electron quantization in (2.8), (2.9). Quasihole quantization will emerge in the quantum theory, as we shall see shortly.

3. A model for finite number of electrons

3.1 The Chern-Simons finite matrix model

We come now to the problem of describing quantum Hall states of finite extent consisting of N electrons. Obviously the coordinates X_a now have to be represented by finite $N \times N$ matrices. The action (2.1), however, and the equation (2.3) to which it leads, are inconsistent for finite matrices, and a modified action must be written which still captures the physical features of the quantum Hall system. Such an action in fact exists [36]. For our purposes we take

$$S = \int dt \frac{B}{2} \mathrm{Tr} \left\{ \epsilon_{ab} (\dot{X}_a + i[A_0, X_a]) X_b + 2\theta A_0 - \omega X_a^2 \right\} + \Psi^\dagger (i\dot{\Psi} - A_0 \Psi) \,. \quad (3.1)$$

It has the same form as the planar CS action, but with two extra terms. The first involves Ψ, a complex N-vector that transforms in the fundamental of the gauge group U(N):

$$X_a \to U X_a U^{-1} \,, \qquad \Psi \to U \Psi \,. \quad (3.2)$$

Its action is a covariant kinetic term similar to a complex scalar fermion. We shall, however, quantize it as a boson; this is perfectly consistent, since there is no spatial kinetic term that would lead to a negative Dirac sea and the usual inconsistencies of first-order bosonic actions.

The term proportional to ω serves as a spatial regulator: since we will be describing a finite number of electrons, there is nothing to keep them localized anywhere in the plane. We added a confining harmonic potential which serves as a 'box' to keep the particles near the origin.

We can again impose the A_0 equation of motion as a Gauss constraint and then put $A_0 = 0$. In our case it reads

$$G \equiv -iB[X_1, X_2] + \Psi \Psi^\dagger - B\theta = 0 \,. \quad (3.3)$$

Taking the trace of the above equation gives

$$\Psi^\dagger \Psi = N B \theta \,. \quad (3.4)$$

The equation of motion for Ψ in the $A_0 = 0$ gauge is $\dot{\Psi} = 0$. So we can take it to be

$$\Psi = \sqrt{NB} \, |v\rangle \,, \quad (3.5)$$

where $|v\rangle$ is a constant vector of unit length. The traceless part of (3.3) reads

$$[X_1, X_2] = i\theta \, (1 - N |v\rangle\langle v|) \,. \quad (3.6)$$

This is similar to (2.3) for the infinite plane case, with an extra projection operator. Using the residual gauge freedom under time-independent unitary transformations, we can rotate $|v\rangle$ to the form $|v\rangle = (0, \ldots 0, 1)$. The above commutator then takes the form $i\theta \, \mathrm{diag} \, (1, \ldots, 1, 1-N)$ which is the 'minimal' deformation of the planar result (2.3) which has a vanishing trace.

In the membrane picture, Ψ is like a boundary term. Its role is to absorb the 'anomaly' of the commutator $[X_1, X_2]$, very much like the case of a boundary field theory required to absorb the anomaly of a bulk CS field theory.

The equations of motion for X_a read

$$\dot{X}_a + \omega \epsilon_{ab} X_b = 0\,. \tag{3.7}$$

This is just a matrix harmonic oscillator. It is solved by

$$X_1 + iX_2 = e^{i\omega t} A\,, \tag{3.8}$$

where A is any $N \times N$ matrix satisfying the constraint

$$[A, A^\dagger] = 2\theta(1 - N|v\rangle\langle v|)\,. \tag{3.9}$$

The classical states of this theory are given by the set of matrices $A = X_1 + iX_2$ satisfying (3.9) or (3.6). We can easily find them by choosing a basis in which one of the X's is diagonal, say, X_1. Then the commutator $[X_1, X_2]$ is purely off-diagonal and the components of the vector $|v\rangle$ must satisfy $|v_n|^2 = 1/N$. We can use the residual $\mathrm{U}(1)^N$ gauge freedom to choose the phases of v_n so that $v_n = 1/\sqrt{N}$. So we get

$$(X_1)_{mn} = x_n \delta_{mn}\,, \qquad (X_2)_{mn} = y_n \delta_{mn} + \frac{i\theta}{x_m - x_n}(1 - \delta_{mn})\,. \tag{3.10}$$

The solution is parametrized by the N eigenvalues if X_1, x_n, and the N diagonal elements of X_2, y_n.

3.2 Quantum Hall 'droplet' vacuum

Not all solutions found above correspond to quantum Hall fluids. In fact, choosing all x_n and y_n much bigger than $\sqrt{\theta}$ and not too close to each other, both X_1 and X_2 become almost diagonal; they represent N electrons scattered in positions (x_n, y_n) on the plane and performing rotational motion around the origin with angular velocity ω. This is the familiar motion of charged particles in a magnetic field along lines of equal potential when their proper kinetic term is negligible. Quantum Hall states will form when particles coalesce near the origin, that is, for states of low energy.

To find the ground state, we must minimize the potential

$$V = \frac{B\omega}{2}\mathrm{Tr}(X_1^2 + X_2^2) = \frac{B\omega}{2}\mathrm{Tr}(A^\dagger A) \tag{3.11}$$

while imposing the constraint (3.6) or (3.9). This can be implemented with a matrix Lagrange multiplier Λ and we obtain

$$A = [\Lambda, A]\,, \qquad \text{or} \qquad X_a = i\epsilon_{ab}[\Lambda, X_b]\,. \tag{3.12}$$

This is reminiscent of canonical commutation relations for a quantum harmonic oscillator, with Λ playing the role of the hamiltonian. We are led to the solution

$$A = \sqrt{2\theta}\sum_{n=0}^{N-1}\sqrt{n}|n-1\rangle\langle n|\,, \qquad \Lambda = \sum_{n=0}^{N-1} n|n\rangle\langle n|\,, \qquad |v\rangle = |N-1\rangle\,. \tag{3.13}$$

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This is essentially a quantum harmonic oscillator and hamiltonian projected to the lowest N energy eigenstates. It is easy to check that the above satisfies both (3.9) and (3.12). Its physical interpretation is clear: it represents a circular quantum Hall 'droplet' of radius $\sqrt{2N\theta}$. Indeed, the radius-squared matrix coordinate R^2 is

$$\begin{aligned} R^2 &= X_1^2 + X_2^2 = A^\dagger A + \frac{1}{2}[a, a^\dagger] \\ &= \sum_{n=0}^{N-2} \theta(2n+1)|n\rangle\langle n| + \theta(N-1)|N-1\rangle\langle N-1|\,. \end{aligned} \tag{3.14}$$

The highest eigenvalue of R^2 is $(2N-1)\theta$. The particle density of this droplet is $\rho_0 = N/(\pi R^2) \sim 1/(2\pi\theta)$ as in the infinite plane case.

The matrices X_a are known and can be explicitly diagonalized in this case. Their eigenvalues are given by the zeros of the N-th Hermite polynomial (times $\sqrt{2\theta}$). In the large-N limit the distribution of these zeros obeys the famous Wigner semi-circle law, with radius $\sqrt{N}$. Since these eigenvalues are interpreted as electron coordinates, this confirms once more the fact that the electrons are evenly distributed on a circle of radius $\sqrt{2N\theta}$.

3.3 Excited states of the model

Excitations of the classical ground state can now be considered. Any perturbation of (3.13) in the form of (3.10) is, of course, some excited state. We shall concentrate, however, on two special types of excitations.

The first is obtained by performing on $A, A^\dagger$ all transformations generated by the infinitesimal transformation

$$A' = A + \sum_{n=0}^{N-1} \epsilon_n (A^\dagger)^n \tag{3.15}$$

with ϵ_n infinitesimal complex parameters. The sum is truncated to $N-1$ since $A^\dagger$ is an $N \times N$ matrix and only its first N powers are independent. It is obvious that (3.9) remains invariant under the above transformation and therefore also under the finite transformations generated by repeated application of (3.15).

If $A, A^\dagger$ were true oscillator operators, these would be canonical (unitary) transformations, that is, gauge transformations that would leave the physical state invariant. For the finite $A, A^\dagger$ in (3.13), however, these are *not* unitary transformations and generate a new state. To understand what is that new state, examine what happens to the 'border' of the circular quantum Hall droplet under this transformation. This is defined by $A^\dagger A \sim 2N\theta$ (for large N). To find the new boundary parametrize $A \sim \sqrt{2N\theta} e^{i\phi}$, with ϕ the polar angle on the plane and calculate $(A^\dagger A)'$. The new boundary in polar coordinates is

$$R'(\phi) = \sqrt{2N\theta} + \sum_{n=-N}^{N} c_n e^{in\phi}\,, \tag{3.16}$$

where the coefficients c_n are

$$c_n = c_{-n}^\dagger = \frac{R^n}{2}\epsilon_{n-1} \qquad (n > 0)\,, \qquad c_0 = 0\,. \tag{3.17}$$

This is an arbitrary area-preserving deformation of the boundary of the droplet, truncated to the lowest N Fourier modes. The above states are, therefore, arbitrary area-preserving boundary excitations of the droplet [37, 38, 39], appropriately truncated to reflect the finite noncommutative nature of the system (the fact that there are only N electrons).

Note that on the plane there is an infinity of area-preserving diffeomorphisms that produce a specific deformation of a given curve. From the droplet point of view, however, these are all gauge equivalent since they deform the outside of the droplet (which is empty) or the inside of it (which is full and thus invariant). The finite theory that we examine has actually broken this infinite gauge freedom, since most of these canonical transformations of $a, a^\dagger$ do not preserve the Gauss constraint (3.9) when applied on $A, A^\dagger$. The transformations (3.15) pick a representative in this class which respects the constraint.

The second class of excitations are the analogs of quasihole and quasiparticle states. States with a quasihole of charge $-q$ at the origin can be written quite explicitly in the form

$$A = \sqrt{2\theta}\left(\sqrt{q}|N-1\rangle\langle 0| + \sum_{n=1}^{N-1}\sqrt{n+q}|n-1\rangle\langle n|\right)\,, \qquad q > 0\,. \tag{3.18}$$

It can be verified that the eigenvalues of $A^\dagger A$ are

$$(A^\dagger A)_n = 2\theta(n+q)\,, \qquad n = 0, 1, \dots N-1 \tag{3.19}$$

so it represents a circular droplet with a circular hole of area $2\pi\theta q$ at the origin, that is, with a charge deficit q. The droplet radius has appropriately swelled, since the total number of particles is always N.

Note that (3.18) stills respects the Gauss constraint (3.9) (with $|v\rangle = |N-1\rangle$) *without* the explicit introduction of any source. So, unlike the infinite plane case, this model contains states representing quasiholes without the need to introduce external sources. What happens is that the hole and the boundary of the droplet together cancel the anomaly of the commutator, the outer boundary part absorbing an amount $N+q$ and the inner (hole) boundary producing an amount q. This possibility did not exist in the infinite plane, where the boundary at infinity was invisible, and an explicit source was needed to nucleate the hole. We view this as an advantage of the finite N matrix model.

Quasiparticle states are a different matter. In fact, there are *no* quasiparticle states with the extra particle number localized anywhere within the droplet. Such states do not belong to the $\nu = 1/B\theta$ Laughlin state. There are quasiparticle states with an *integer* particle number $-q = m$, and the extra m electrons occupying positions *outside* the droplet. The explicit form of these states is not so easy to write. At any rate, it is interesting that the matrix model 'sees' the quantization of the particle number and the inaccessibility of the interior of the quantum Hall state in a natural way.

Having said all that, we are now making the point that *all types of states defined above are the same.* Quasihole and quasiparticle states are nonperturbative boundary excitations of the droplet, while perturbative boundary excitations can be viewed as marginal particle states.

To clarify this point, note that the transformation (3.15) or (3.16) defining infinitesimal boundary excitations has $2N$ real parameters. The general state of the system, as presented in (3.10) also depends on $2N$ parameters (the x_n and y_n). The configuration space is connected, so all states can be reached continuously from the ground state. Therefore, all states can be generated by exponentiating (3.15). This is again a feature of the finite-N model: there is no sharp distinction between 'perturbative' (boundary) and 'soliton' (quasiparticle) states, each being a particular limit of the other.

3.4 Equivalence to the Calogero model

We now make one of the main points of this paper. The model examined above is, in fact, equivalent to the Calogero model [24, 36], an integrable system of N nonrelativistic particles on the line with hamiltonian

$$H = \sum_{n=1}^{N} \left(\frac{\omega}{2B} p_n^2 + \frac{B\omega}{2} x_n^2 \right) + \sum_{n \neq m} \frac{\nu^{-2}}{(x_n - x_m)^2} \,. \tag{3.20}$$

In terms of the parameters of the model, the mass of the particles is B/ω and the coupling constant of the two-body inverse-square potential is ν^{-2}. We refer the reader to [24, 25, 36] for details of the derivation of the connection between the matrix model and the Calogero model. Here we simply state the relevant results and give their connection to quantum Hall quantities.

The positions of the Calogero particles x_n are the eigenvalues of X_1, while the momenta p_n are the diagonal elements of X_2, specifically $p_n = By_n$. The motion of the x_n generated by the hamiltonian (3.20) is compatible with the evolution of the eigenvalues of X_1 as it evolves in time according to (3.8). So the Calogero model gives a one-dimensional perspective of the quantum Hall state by monitoring some effective electron coordinates along X_1 (the eigenvalues of X_1).

The hamiltonian of the Calogero model (3.20) is equal to the potential $V = \frac{1}{2}B\omega \mathrm{Tr} X_a^2$ of the matrix model. Therefore, energy states map between the two models. The ground state is obtained by putting the particles at their static equilibrium positions. Because of their repulsion, they will form a lattice of points lying at the roots of the N-th Hermite polynomial and reproducing the semi-cicle Wigner distribution mentioned before.

Boundary excitations of the quantum Hall droplet correspond to small vibrations around the equilibrium position, that is, sound waves on the lattice. Quasiholes are large-amplitude (nonlinear) oscillations of the particles at a localized region of the lattice. For a quasihole of charge q at the center, on the average q particles near $x = 0$ participate in the oscillation.

Finally, quasiparticles are excitations where one of the particles is isolated outside the ground state distribution (a 'soliton') [40]. As it moves, it 'hits' the distribution on one side and causes a solitary wave of net charge 1 to propagate through the distribution. As the wave reaches the other end of the distribution another particle emerges and gets emitted there, continuing its motion outside the distribution. So a quasiparticle is more or less identified with a Calogero particle, although its role, at different times, is assumed by different Calogero particles, or even by soliton waves within the ground state distribution.

Overall, it is amusing and useful to have the one-dimensional Calogero particle picture of the quantum Hall state and translate properties back-and-forth between the two descriptions. Further connections at the quantum level will be described in subsequent sections.

4. Quantization of the matrix Chern-Simons model

4.1 Gauss law and quantization of the filling fraction

We now come to the question of the quantization of the above matrix model. This is an important issue, since some of the relevant features of the quantum Hall state will only emerge at the quantum level. The quantization has been treated in [36]. We shall repeat here the basic arguments establishing their relevance to the quantum Hall system.

We shall use double brackets for quantum commutators and double kets for quantum states, to distinguish them from matrix commutators and N-vectors.

Quantum mechanically the matrix elements of X_a become operators. Since the lagrangian is first-order in time derivatives, $(X_1)_{mn}$ and $(X_2)_{kl}$ are canonically conjugate:

$$[[(X_1)_{mn}, (X_2)_{kl}]] = \frac{i}{B}\delta_{ml}\delta_{kn} \tag{4.1}$$

or, in terms of $A = X_1 + iX_2$

$$[[A_{mn}, A^\dagger_{kl}]] = \frac{1}{B}\delta_{mk}\delta_{nl}\,. \tag{4.2}$$

The hamiltonian, ordered as $\frac{1}{2}B\omega \mathrm{Tr} A^\dagger A$, is

$$H = \sum_{mn} \frac{1}{2} B\omega A^\dagger_{mn} A_{mn} \,. \tag{4.3}$$

This is just N^2 harmonic oscillators. Further, the components of the vector Ψ_n correspond to N harmonic oscillators. Quantized as bosons, their canonical commutator is

$$[[\Psi_m, \Psi^\dagger_n]] = \delta_{mn} \,. \tag{4.4}$$

So the system is a priori just $N(N+1)$ uncoupled oscillators. What couples the oscillators and reduces the system to effectively $2N$ phase space variables (the planar coordinates of the electrons) is the Gauss law constraint (3.3). In writing it, we in principle encounter operator ordering ambiguities. These are easily fixed, however, by noting that the operator G is the quantum generator of unitary rotations of both X_a and Ψ. Therefore, it must satisfy the commutation relations of the $\mathrm{U}(N)$ algebra. The X-part is an orbital realization of $\mathrm{SU}(N)$ on the manifold of $N \times N$ hermitian matrices. Specifically, expand $X_{1,2}$ and $A, A^\dagger$ in the complete basis of matrices $\{1, T^a\}$ where T^a are the $N^2 - 1$ normalized fundamental $\mathrm{SU}(N)$ generators:

$$X_1 = x_0 + \sum_{a=1}^{N^2-1} x_a T^a \,, \qquad \sqrt{B} A = a_o + \sum_{a=1}^{N^2-1} a_a T^a \tag{4.5}$$

x_a, a_a are scalar operators. Then, by (4.1), (4.2) the corresponding components of BX_2 are the conjugate operators $-i\partial/\partial x_a$, while $a_a, a^\dagger_a$ are harmonic oscillator operators. We can write the components of the matrix commutator $G_X = -iB[X_1, X_2]$ in G in the following ordering

$$\begin{aligned} G^a_X &= -i f^{abc} x_b \frac{\partial}{\partial x_a} \\ &= -i(A^\dagger_{mk} A_{nk} - A^\dagger_{nk} A_{mk}) \\ &= -i a^\dagger_b f^{abc} a_c \,, \end{aligned} \tag{4.6}$$

where f^{abc} are the structure constants of $\mathrm{SU}(N)$. Similarly, expressing $G_\Psi = \Psi\Psi^\dagger$ in the $\mathrm{SU}(N)$ basis of matrices, we write its components in the ordering

$$G^a_\Psi = \Psi^\dagger_m T^a_{mn} \Psi_n \,. \tag{4.7}$$

The operators above, with the specific normal ordering, indeed satisfy the $\mathrm{SU}(N)$ algebra. The expression of G^a_X in terms of x_a is like an angular momentum. The expression of G^a_Ψ in terms of the oscillators Ψ_i and of G^a_X in terms of the oscillators a_a is the well-known Jordan-Wigner realization of the $\mathrm{SU}(N)$ algebra in the Fock space of bosonic oscillators. Specifically, let $R^a_{\alpha\beta}$ be the matrix elements of the generators

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of SU(N) in any representation of dimension d_R, and $a_\alpha, a_\alpha^\dagger$ a set of d_R mutually commuting oscillators. Then the operators

$$G^a = a_\alpha^\dagger R^a_{\alpha\beta} a_\beta \tag{4.8}$$

satisfy the SU(N) algebra. The Fock space of the oscillators contains all the symmetric tensor products of R-representations of SU(N); the total number operator of the oscillators identifies the number of R components in the specific symmetric product. The expressions for G^a_Ψ and G^a_X are specific cases of the above construction for R^a the fundamental (T^a) or the adjoint ($-if^a$) representation respectively.

So, the traceless part of the Gauss law (3.3) becomes

$$(G^a_X + G^a_\Psi)|\mathrm{phys}\rangle\rangle = 0\,, \tag{4.9}$$

where $|\mathrm{phys}\rangle\rangle$ denotes the physical quantum states of the model. The trace part, on the other hand, expresses the fact that the total U(1) charge of the model must vanish. It reads

$$(\Psi_n^\dagger \Psi_n - NB\theta)|\mathrm{phys}\rangle\rangle = 0\,. \tag{4.10}$$

We are now set to derive the first nontrivial quantum mechanical implication: the inverse-filling fraction is quantized to integer values. To see this, first notice that the first term in (4.10) is nothing but the total number operator for the oscillators Ψ_n and is obviously an integer. So we immediately conclude that $NB\theta$ must be quantized to an integer.

However, this is not the whole story. Let us look again at the SU(N) Gauss law (4.9). It tells us that physical states must be in a singlet representation of G^a. The orbital part G^a_X, however, realizes only representations arising out of products of the adjoint, and therefore it contains only irreps whose total number of boxes in their Young tableau is an integer multiple of N. Alternatively, the U(1) and Z_N part of U is invisible in the transformation $X_a \to U X_a U^{-1}$ and thus the Z_N charge of the operator realizing this transformation on states must vanish. (For instance, for $N = 2$, G^a is the usual orbital angular momentum in 3 dimensions which cannot be half-integer.)

Since physical states are invariant under the sum of G_X and G_Ψ, the representations of G_Ψ and G_X must be conjugate to each other so that their product contain the singlet. Therefore, the irreps of G_Ψ must also have a number of boxes which is a multiple of N. The oscillator realization (4.10) contains all the symmetric irreps of SU(N), whose Young tableau consists of a single row. The number of boxes equals the total number operator of the oscillators $\Psi_n^\dagger \Psi_n$. So we conclude that $NB\theta$ must be an integer multiple of N [36], that is,

$$B\theta = k\,, \qquad k = \text{integer}\,. \tag{4.11}$$

The above effect has a purely group theoretic origin. The same effect, however, can be recovered using topological considerations, by demanding invariance of the quantum action $\exp(iS)$ under gauge $\mathrm{U}(N)$ transformations with a nontrivial winding in the temporal direction [36]. The $\mathrm{U}(1)$ part of $\mathrm{U}(t)$ makes a nonzero contribution to the one-dimensional Chern-Simons term $\mathrm{Tr}A_0$, and we recover (4.11). This is the finite-N counterpart of the level quantization for the noncommutative Chern-Simons term [19], namely

$$4\pi\lambda = \text{integer}\,. \tag{4.12}$$

By (2.4) this is equivalent to (4.11).

By reducing the model to the dynamics of the eigenvalues of X_1 we recover a quantum Calogero model with hamiltonian

$$H = \sum_{n=1}^{N}\left(\frac{\omega}{2B}p_n^2 + \frac{B\omega}{2}x_n^2\right) + \sum_{n\neq m}\frac{k(k+1)}{(x_n - x_m)^2}\,. \tag{4.13}$$

Note the shift of the coupling constant from k^2 to $k(k+1)$ compared to the classical case. This is a quantum reordering effect which results in the shift of ν^{-1} from k to $k+1 \equiv n$. The above model is, in fact, perfectly well-defined even for fractional values of ν^{-1}, while the matrix model that generated it requires quantization. This is due to the fact that, by embedding the particle system in the matrix model, we have augmented its particle permutation symmetry S_N to general $\mathrm{U}(N)$ transformations; while the smaller symmetry S_N is always well-defined, the larger $\mathrm{U}(N)$ symmetry becomes anomalous unless ν^{-1} is quantized.

4.2 Quantum states

We can now examine the quantum states of this theory. The results are, again, available and all we need to do is interpret them.

The quantum states of the model are simply states in the Fock space of a collection of oscillators. The total energy is the energy carried by the N^2 oscillators A_{mn} or a_a. We must also impose the constraint (4.9) and (4.10) on the Fock states. Overall, this becomes a combinatorics group theory problem which is in principle doable, although quite tedious.

Fortunately, we do not need to go through it here. The quantization of this model is known and achieves its most intuitive description in terms of the states of the corresponding Calogero model. We explain how.

Let us work in the X_1 representation, X_2 being its canonical momentum. Writing $X_1 = U\Lambda_1 U^{-1}$ with $\Lambda = \mathrm{diag}\,\{x_i\}$ being its eigenvalues, we can view the state of the system as a wavefunction of U and x_n. The gauge generator G_X^a appearing in the Gauss law (4.9) is actually the conjugate momentum to the variables U. Due to the Gauss law, the angular degrees of freedom U are constrained to be in a specific angular momentum state, determined by the representation of $\mathrm{SU}(N)$ carried by the Ψ_n. From the discussion of the previous section, we understand that this is the completely symmetric representation with $kN = (n-1)N$ boxes in the Young tableau. So the dynamics of U are completely fixed, and it suffices to consider the states of the eigenvalues. These are described by the states of the quantum Calogero model. The hamiltonian of the Calogero model corresponds to the matrix potential $V = \frac{1}{2}B\omega \mathrm{Tr} X_a^2$, which contains all the relevant information for the system.

Calogero energy eigenstates are expressed in terms of N positive, integer 'quasi-occupation numbers' n_j (quasinumbers, for short), with the property

$$n_j - j_{j-1} \geq n = \frac{1}{\nu}\,, \qquad j = 1, \ldots N\,. \tag{4.14}$$

In terms of the n_j the spectrum becomes identical to the spectrum of N independent harmonic oscillators

$$E = \sum_{j=1}^{N} E_j = \sum_{j=1}^{N} \omega \left(n_j + \frac{1}{2}\right). \tag{4.15}$$

The constraint (4.14) means that the n_j cannot be packed closer than $n = \nu^{-1}$, so they have a 'statistical repulsion' of order n. For filling fraction $\nu = 1$ these are ordinary fermions, while for $\nu^{-1} = n > 1$ they behave as particles with an enhanced exclusion principle.

The scattering phase shift between Calogero particles is $\exp(i\pi/\nu)$. So, in terms of the phase that their wavefunction picks upon exchanging them, they look like fermions for odd n and bosons for even n [26]. Since the underlying particles (electrons) must be fermions, we should pick n odd.

The energy 'eigenvalues' E_j are the quantum analogs of the eigenvalues of the matrix $\frac{1}{2}B\omega X_a^2$. The radial positions R_j are determined by

$$\frac{1}{2}B\omega R_j^2 = E_j \qquad \rightarrow \qquad R_j^2 = \frac{2n_j + 1}{B}\,. \tag{4.16}$$

So the quasinumbers $2n_j + 1$ determine the radial positions of electrons. The ground state values are the smallest non-negative integers satisfying (4.14)

$$n_{j,gs} = n(j-1)\,, \qquad j = 1, \ldots N\,. \tag{4.17}$$

They form a 'Fermi sea' but with a density of states dilated by a factor ν compared to standard fermions. This state reproduces the circular quantum Hall droplet. Its radius maps to the Fermi level, $R \sim \sqrt{(2n_{N,gs}+1)/B} \sim \sqrt{2N\theta}$.

Quasiparticle and quasihole states are identified in a way analogous to particles and holes of a Fermi sea. A quasiparticle state is obtained by peeling a 'particle' from the surface of the sea (quasinumber $n_{N,gs}$) and putting it to a higher value $n'_N > n(N-1)$. This corresponds to an electron in a rotationally invariant state at radial position $R' \sim \sqrt{2(n'_N + 1)/B}$. Successive particles can be excited this way. The particle number is obviously quantized to an integer (the number of excited quasinumbers) and we can only place them outside the quantum Hall droplet.

Quasiholes are somewhat subtler: they correspond to the minimal excitations of the ground state *inside* the quantum Hall droplet. This can be achieved by leaving all quasinumber n_j for $j \leq k$ unchanged, and increasing all n_j, $j > k$ by one

$$\begin{aligned} n_j &= n(j-1) \qquad j \leq k \\ &= n(j-1)+1 \qquad k < j \leq N\,. \end{aligned} \tag{4.18}$$

This increases the gap between n_k and n_{k+1} to $n+1$ and creates a minimal 'hole.'

This hole has a particle number $-q = -1/n = -\nu$. To see it, consider removing a particle altogether from quasinumber n_k. This would create a gap of $2n$ between n_{k-1} and n_{k+1}. The extra gap n can be considered as arising out of the formation of n holes (increasing n_j for $j \geq k$ n times). Thus the absence of a particle corresponds to n holes. We therefore obtain the important result that the quasihole charge is naturally quantized to units of

$$q_h = \nu = \frac{1}{n} \tag{4.19}$$

in accordance with Laughlin theory.

We conclude by stressing once more that there is no fundamental distinction between particles and holes for finite N. A particle can be considered as a nonperturbative excitation of many holes near the Fermi level, while a hole can be viewed as a coherent state of many particles of minimal excitation.

5. Epilogue

We have proposed a finite matrix Chern-Simons model and presented strong evidence that it describes a fractional quantum Hall droplet. We identified classical and quantum states of this model and related them to corresponding fractional quantum Hall states. The quantization of the inverse filling fraction and, importantly, the quasihole charge quantization emerged as quantum mechanical consequences of this model.

The quantizations of the two parameters had a rather different origin. We can summarize here the basic meaning of each:

Quantization of the inverse filling fraction is basically angular momentum quantization. The matrix commutator of $[X_1, X_2]$ is an orbital angular momentum in the compact space of the angular parameters of the matrices, and it must be quantized.

Quantization of the quasihole charge is nothing but harmonic oscillator quantization. Quasiholes are simply individual quanta of the oscillators A_{mn}. The square of the radial coordinate $R^2 = X_1^2 + X_2^2$ is basically a harmonic oscillator. $\sqrt{B}X_1$ and $\sqrt{B}X_2$ are canonically conjugate, so the quanta of R^2 are $2/B$. Each quantum increases R^2 by $2/B$ and so it increases the area by $2\pi/B$. This creates a charge deficit q equal to the area times the ground state density $q = (2\pi/B)\cdot(1/2\pi\theta) = 1/\theta B = \nu$. So the fundamental quasihole charge is ν.

We further pointed out that this model is essentially equivalent to the Calogero integrable model, providing another link between the Calogero and Hall systems.

The most important open question is the existence of a phase transition of the system into a Wigner crystal at low filling fractions. In fact, we have the complete quantum mechanics of the model, yet we found no indication of such a phase transition. It is plausible that the phase transition would only emerge at the large-N limit, but the (known) large-N behavior of the model does not manifest such a transition.

A more likely scenario is that the dynamics of the particles will drive such a phase transition, rather than the properties of the Laughlin state itself. The electron interactions would presumably be an important ingredient of the mechanism [17]. In order to probe this question we would need to formulate a model which includes mutual particle interactions. Such a model is presently lacking.

In [13] is was argued that a signal for the broken phase would be the spontaneous breaking of the X-space area-preserving diffeomorphism invariance. The vacuum of the infinite plane Chern-Simons theory is invariant under all these transformations, since they amount to canonical transformations of the Heisenberg algebra $[X_1, X_2] = i\theta$ which are gauge transformations. The matrix model proposed here breaks this invariance, as we pointed out in section 3.3. This is due, however, to the finite extent of the model, and specifically to the presence of the boundary of the droplet, and it is not an indication of a phase transition.

Although the above model gives an accurate and compelling description of the fractional quantum Hall states and many of their properties, it still leaves unanswered some questions that could in principle be addressed in the quantum mechanical many-body language. In particular, we do not have a good description for quantities like the electron density and correlation functions. Clearly the dictionary between the two systems needs to be expanded and completed. Functional descriptions of the quantum phase space (viewed as a noncommutative space) such as Wigner functions (see, e.g., [41]) may prove useful tools. The above questions are left for future research.

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